Translating English Sentences

There are many reasons to translate English sentences into expressions involving propositional variables and logical connectives. In particular, English (and every other human language) is often ambiguous. Translating sentences into compound statements (and other types of logical expressions, which we will introduce later in this chapter) removes the ambiguity. Note that this may involve making a set of reasonable assumptions based on the intended meaning of the sentence. Moreover, once we have translated sentences from English into logical expressions we can analyze these logical expressions to determine their truth values, we can manipulate them, and we can use rules of inference (which are discussed in Section 1.6) to reason about them.

To illustrate the process of translating an English sentence into a logical expression, consider Examples 1 and 2.

Example 1
How can this English sentence be translated into a logical expression?

“You can access the Internet from campus only if you are a computer science major or you are not a freshman.”

Solution: There are many ways to translate this sentence into a logical expression. Although it is possible to represent the sentence by a single propositional variable, such as $p$, this would not be useful when analyzing its meaning or reasoning with it. Instead, we will use propositional variables to represent each sentence part and determine the appropriate logical connectives between them. In particular, we let $a$, $c$, and $f$ represent “You can access the Internet from campus,” “You are a computer science major,” and “You are a freshman,” respectively. Noting that “only if” is one way a conditional statement can be expressed, this sentence can be represented as

$$a \rightarrow (c \lor \neg f).$$

Example 2
How can this English sentence be translated into a logical expression?

“You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.”

Solution: Let $q$, $r$, and $s$ represent “You can ride the roller coaster,” “You are under 4 feet tall,” and “You are older than 16 years old,” respectively. Then the sentence can be translated to

$$(r \land \neg s) \rightarrow \neg q.$$

Of course, there are other ways to represent the original sentence as a logical expression, but the one we have used should meet our needs.
Propositional Equivalences

Introduction

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value. Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments. Note that we will use the term “compound proposition” to refer to an expression formed from propositional variables using logical operators, such as \( p \land q \).

We begin our discussion with a classification of compound propositions according to their possible truth values.

**DEFINITION 1**

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology. A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Tautologies and contradictions are often important in mathematical reasoning. Example 1 illustrates these types of compound propositions.

**Example 1**

We can construct examples of tautologies and contradictions using just one propositional variable. Consider the truth tables of \( p \lor \neg p \) and \( p \land \neg p \), shown in Table 1. Because \( p \lor \neg p \) is always true, it is a tautology. Because \( p \land \neg p \) is always false, it is a contradiction.

Logical Equivalences

Compound propositions that have the same truth values in all possible cases are called logically equivalent. We can also define this notion as follows.

**DEFINITION 2**

The compound propositions \( p \) and \( q \) are called logically equivalent if \( p \leftrightarrow q \) is a tautology. The notation \( p \equiv q \) denotes that \( p \) and \( q \) are logically equivalent.

**Remark:** The symbol \( \equiv \) is not a logical connective, and \( p \equiv q \) is not a compound proposition but rather is the statement that \( p \leftrightarrow q \) is a tautology. The symbol \( \Leftrightarrow \) is sometimes used instead of \( \equiv \) to denote logical equivalence.

One way to determine whether two compound propositions are equivalent is to use a truth table. In particular, the compound propositions \( p \) and \( q \) are equivalent if and only if the columns
TABLE 1 Examples of a Tautology and a Contradiction.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
<th>$p \lor \neg p$</th>
<th>$p \land \neg p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

TABLE 2 De Morgan’s Laws.

$\neg(p \land q) \equiv \neg p \lor \neg q$

$\neg(p \lor q) \equiv \neg p \land \neg q$

giving their truth values agree. Example 2 illustrates this method to establish an extremely important and useful logical equivalence, namely, that of $\neg(p \lor q)$ with $\neg p \land \neg q$. This logical equivalence is one of the two De Morgan laws, shown in Table 2, named after the English mathematician Augustus De Morgan, of the mid-nineteenth century.

Example 2
Show that $\neg(p \land q)$ and $\neg p \land \neg q$ are logically equivalent.

Solution: The truth tables for these compound propositions are displayed in Table 3. Because the truth values of the compound propositions $\neg(p \lor q)$ and $\neg p \land \neg q$ agree for all possible combinations of the truth values of $p$ and $q$, it follows that $\neg(p \lor q) \leftrightarrow (\neg p \land \neg q)$ is a tautology and that these compound propositions are logically equivalent.

TABLE 3 Truth Tables for $\neg(p \lor q)$ and $\neg p \land \neg q$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \lor q$</th>
<th>$\neg(p \lor q)$</th>
<th>$\neg p$</th>
<th>$\neg q$</th>
<th>$\neg p \land \neg q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
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<tr>
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<td>F</td>
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<td>T</td>
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<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Example 3
Show that $p \to q$ and $\neg p \lor q$ are logically equivalent.

Solution: We construct the truth table for these compound propositions in Table 4. Because the truth values of $\neg p \lor q$ and $p \to q$ agree, they are logically equivalent.
# TABLE 4  Truth Tables for \( \neg p \lor q \) and \( p \rightarrow q \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg p )</th>
<th>( \neg p \lor q )</th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
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<tr>
<td>T</td>
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<td>F</td>
<td>F</td>
<td>T</td>
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</tbody>
</table>

We will now establish a logical equivalence of two compound propositions involving three different propositional variables \( p \), \( q \), and \( r \). To use a truth table to establish such a logical equivalence, we need eight rows, one for each possible combination of truth values of these three variables. We symbolically represent these combinations by listing the truth values of \( p \), \( q \), and \( r \), respectively. These eight combinations of truth values are TTT, TTF, TFT, TFF, FFT, FFT, FTF, FFT, and FFF; we use this order when we display the rows of the truth table. Note that we need to double the number of rows in the truth tables we use to show that compound propositions are equivalent for each additional propositional variable, so that 16 rows are needed to establish the logical equivalence of two compound propositions involving four propositional variables, and so on. In general, \( 2^n \) rows are required if a compound proposition involves \( n \) propositional variables.

# TABLE 5  A Demonstration That \( p \lor (q \land r) \) and \( (p \lor q) \land (p \lor r) \) Are Logically Equivalent.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
<th>( q \land r )</th>
<th>( p \lor (q \land r) )</th>
<th>( p \lor q )</th>
<th>( p \lor r )</th>
<th>( (p \lor q) \land (p \lor r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<td>T</td>
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<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

**Example 4**

Show that \( p \lor (q \land r) \) and \( (p \lor q) \land (p \lor r) \) are logically equivalent. This is the *distributive law* of disjunction over conjunction.

**Solution:** We construct the truth table for these compound propositions in Table 5. Because the truth values of \( p \lor (q \land r) \) and \( (p \lor q) \land (p \lor r) \) agree, these compound propositions are logically equivalent.
Table 6 contains some important equivalences. In these equivalences, $T$ denotes the compound proposition that is always true and $F$ denotes the compound proposition that is always false.

**TABLE 6** Logical Equivalences.

<table>
<thead>
<tr>
<th>Equivalence</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \land T \equiv p$</td>
<td>Identity laws</td>
</tr>
<tr>
<td>$p \lor F \equiv p$</td>
<td></td>
</tr>
<tr>
<td>$p \lor T \equiv T$</td>
<td>Domination laws</td>
</tr>
<tr>
<td>$p \land F \equiv F$</td>
<td></td>
</tr>
<tr>
<td>$p \lor p \equiv p$</td>
<td>Idempotent laws</td>
</tr>
<tr>
<td>$p \land p \equiv p$</td>
<td></td>
</tr>
<tr>
<td>$\neg(\neg p) \equiv p$</td>
<td>Double negation law</td>
</tr>
<tr>
<td>$p \lor q \equiv q \lor p$</td>
<td></td>
</tr>
<tr>
<td>$p \land q \equiv q \land p$</td>
<td></td>
</tr>
<tr>
<td>$(p \lor q) \lor r \equiv p \lor (q \lor r)$</td>
<td>Associative laws</td>
</tr>
<tr>
<td>$(p \land q) \land r \equiv p \land (q \land r)$</td>
<td></td>
</tr>
<tr>
<td>$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$</td>
<td>Distributive laws</td>
</tr>
<tr>
<td>$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$</td>
<td></td>
</tr>
<tr>
<td>$\neg(p \land q) \equiv \neg p \lor \neg q$</td>
<td>De Morgan’s laws</td>
</tr>
<tr>
<td>$\neg(p \lor q) \equiv \neg p \land \neg q$</td>
<td></td>
</tr>
<tr>
<td>$p \land (p \lor q) \equiv p$</td>
<td>Absorption laws</td>
</tr>
<tr>
<td>$p \lor (p \land q) \equiv p$</td>
<td></td>
</tr>
<tr>
<td>$p \lor \neg p \equiv T$</td>
<td>Negation laws</td>
</tr>
<tr>
<td>$p \land \neg p \equiv F$</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 7** Logical Equivalences Involving Conditional Statements.

<table>
<thead>
<tr>
<th>Equivalence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \to q \equiv \neg p \lor q$</td>
</tr>
<tr>
<td>$p \to q \equiv \neg q \to \neg p$</td>
</tr>
<tr>
<td>$p \lor q \equiv \neg p \to q$</td>
</tr>
<tr>
<td>$p \land q \equiv \neg(p \to \neg q)$</td>
</tr>
<tr>
<td>$\neg(p \to q) \equiv p \land \neg q$</td>
</tr>
<tr>
<td>$(p \to q) \land (p \to r) \equiv p \to (q \land r)$</td>
</tr>
<tr>
<td>$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$</td>
</tr>
<tr>
<td>$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$</td>
</tr>
<tr>
<td>$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$</td>
</tr>
</tbody>
</table>

**TABLE 8** Logical Equivalences Involving Biconditional Statements.

<table>
<thead>
<tr>
<th>Equivalence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \iff q \equiv (p \to q) \land (q \to p)$</td>
</tr>
<tr>
<td>$p \iff q \equiv \neg p \iff \neg q$</td>
</tr>
<tr>
<td>$p \iff q \equiv (p \land q) \lor (\neg p \land \neg q)$</td>
</tr>
<tr>
<td>$\neg(p \iff q) \equiv p \iff \neg q$</td>
</tr>
</tbody>
</table>
false. We also display some useful equivalences for compound propositions involving conditional statements and biconditional statements in Tables 7 and 8, respectively. The reader is asked to verify the equivalences in Tables 6–8 in the exercises.

The associative law for disjunction shows that the expression \( p \lor q \lor r \) is well defined, in the sense that it does not matter whether we first take the disjunction of \( p \) with \( q \) and then the disjunction of \( p \lor q \) with \( r \), or if we first take the disjunction of \( q \) and \( r \) and then take the disjunction of \( p \) with \( q \lor r \). Similarly, the expression \( p \land q \land r \) is well defined. By extending this reasoning, it follows that \( p_1 \lor p_2 \lor \cdots \lor p_n \) and \( p_1 \land p_2 \land \cdots \land p_n \) are well defined whenever \( p_1, p_2, \ldots, p_n \) are propositions.

Furthermore, note that De Morgan’s laws extend to

\[
\neg(p_1 \lor p_2 \lor \cdots \lor p_n) \equiv \neg p_1 \land \neg p_2 \land \cdots \land \neg p_n
\]

And

\[
\neg(p_1 \land p_2 \land \cdots \land p_n) \equiv \neg p_1 \lor \neg p_2 \lor \cdots \lor \neg p_n.
\]

We will sometimes use the notation \( \bigvee_{j=1}^n p_j \) for \( p_1 \lor p_2 \lor \cdots \lor p_n \) and \( \bigwedge_{j=1}^n p_j \) for \( p_1 \land p_2 \land \cdots \land p_n \). Using this notation, the extended version of De Morgan’s laws can be written concisely as \( \neg \left( \bigvee_{j=1}^n p_j \right) \equiv \bigwedge_{j=1}^n \neg p_j \) and \( \neg \left( \bigwedge_{j=1}^n p_j \right) \equiv \bigvee_{j=1}^n \neg p_j \). (Methods for proving these identities will be given in Section 5.1.)

**Example 5**

**Using De Morgan’s Laws**

The two logical equivalences known as De Morgan’s laws are particularly important. They tell us how to negate conjunctions and how to negate disjunctions. In particular, the equivalence \( \neg(p \lor q) \equiv \neg p \land \neg q \) tells us that the negation of a disjunction is formed by taking the conjunction of the negations of the component propositions. Similarly, the equivalence \( \neg(p \land q) \equiv \neg p \lor \neg q \) tells us that the negation of a conjunction is formed by taking the disjunction of the negations of the component propositions. Example 5 illustrates the use of De Morgan’s laws.

Use De Morgan’s laws to express the negations of “Miguel has a cellphone and he has a laptop computer” and “Heather will go to the concert or Steve will go to the concert.”

**Solution:** Let \( p \) be “Miguel has a cellphone” and \( q \) be “Miguel has a laptop computer.” Then “Miguel has a cellphone and he has a laptop computer” can be represented by \( p \land q \). By the first of De Morgan’s laws, \( \neg(p \land q) \) is equivalent to \( \neg p \lor \neg q \). Consequently, we can express the negation of our original statement as “Miguel does not have a cellphone or he does not have a laptop computer.”

Let \( r \) be “Heather will go to the concert” and \( s \) be “Steve will go to the concert.” Then “Heather will go to the concert or Steve will go to the concert” can be represented by \( r \lor s \). By the second of De Morgan’s laws, \( \neg(r \lor s) \) is equivalent to \( \neg r \land \neg s \). Consequently, we can express the negation of our original statement as “Heather will not go to the concert and Steve will not go to the concert.”
Exercises

1. Use truth tables to verify these equivalences.
   a) \( p \land T \equiv p \)
   b) \( p \lor F \equiv p \)
   c) \( p \land F \equiv F \)
   d) \( p \lor T \equiv T \)
   e) \( p \lor p \equiv p \)
   f) \( p \land p \equiv p \)
   g) \( \neg(p \land q) \equiv \neg p \lor \neg q \).

2. Show that \( \neg(\neg p) \) and \( p \) are logically equivalent.

3. Use truth tables to verify the commutative laws.
   a) \( p \lor q \equiv q \lor p \)
   b) \( p \land q \equiv q \land p \).

4. Use truth tables to verify the associative laws.
   a) \( (p \lor q) \lor r \equiv p \lor (q \lor r) \).
   b) \( (p \land q) \land r \equiv p \land (q \land r) \).

5. Use a truth table to verify the distributive law.
   \( p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \).

6. Use a truth table to verify the first De Morgan law.
   \( \neg(p \land q) \equiv \neg p \lor \neg q \).

7. Use De Morgan’s laws to find the negation of each of the following statements.
   a) Jan is rich and happy.
   b) Carlos will bicycle or run tomorrow.

8. Use De Morgan’s laws to find the negation of each of the following statements.
   a) Kwame will take a job in industry or go to graduate school.
   b) Yoshiko knows Java and calculus.
   c) James is young and strong.
   d) Rita will move to Oregon or Washington.

9. Show that each of these conditional statements is a tautology by using truth tables.
   a) \( p \land (p \land q) \rightarrow p \)
   b) \( p \rightarrow (p \lor q) \)
   c) \( \neg p \rightarrow (p \rightarrow q) \)
   d) \( (p \land q) \rightarrow (p \rightarrow q) \)
   e) \( \neg(p \rightarrow q) \rightarrow p \)
   f) \( (p \land q) \rightarrow q \).

10. Show that each of these conditional statements is a tautology by using truth tables.
    a) \( \neg(p \land (p \lor q)) \rightarrow q \)
    b) \( [(p \rightarrow q) \land (q \rightarrow r)] \rightarrow (p \rightarrow r) \)
    c) \( [p \land (p \rightarrow q)] \rightarrow q \)
    d) \( [(p \lor q) \land (p \rightarrow r) \land (q \rightarrow r)] \rightarrow r \).

11. Show that each conditional statement in Exercise 9 is a tautology without using truth tables.

12. Show that each conditional statement in Exercise 10 is a tautology without using truth tables.

13. Use truth tables to verify the absorption laws.
    a) \( p \lor (p \land q) \equiv p \)
    b) \( p \land (p \lor q) \equiv p \)

14. Determine whether \( \neg(p \land (p \rightarrow q)) \rightarrow \neg q \) is a tautology.

15. Determine whether \( \neg(q \land (p \rightarrow q)) \rightarrow \neg p \) is a tautology.

Each of Exercises 16–28 asks you to show that two compound propositions are logically equivalent. To do this, either show that both sides are true, or that both sides are false, for exactly the same combinations of truth values of the propositional variables in these expressions (whichever is easier).

16. Show that \( p \leftrightarrow q \) and \( p \land q \lor \neg(p \land \neg q) \) are logically equivalent.

17. Show that \( \neg(p \land q) \land p \equiv \neg q \) and \( p \leftrightarrow \neg q \) are logically equivalent.

18. Show that \( p \rightarrow q \) and \( \neg q \rightarrow \neg p \) are logically equivalent.

19. Show that \( \neg(p \land q) \land p \equiv \neg q \) are logically equivalent.

20. Show that \( p \rightarrow q \land p \equiv \neg q \) are logically equivalent.

21. Show that \( \neg(p \land q) \land \neg p \equiv q \) are logically equivalent.

22. Show that \( (p \rightarrow q) \land (p \rightarrow r) \) and \( p \rightarrow (q \land r) \) are logically equivalent.

23. Show that \( (p \rightarrow q) \land (q \rightarrow r) \) and \( p \rightarrow (q \land r) \) are logically equivalent.

24. Show that \( (p \rightarrow q) \lor (p \rightarrow r) \) and \( p \rightarrow (q \lor r) \) are logically equivalent.

25. Show that \( (p 

26. Show that \( \neg p \land (q \rightarrow r) \land q \lor r \) are logically equivalent.

27. Show that \( p \leftrightarrow q \land (p \rightarrow q) \land (q \rightarrow p) \) are logically equivalent.

28. Show that \( p \leftrightarrow q \land \neg p \leftrightarrow \neg q \) are logically equivalent.
44. Show that \( \neg \) and \( \land \) form a functionally complete collection of logical operators. [Hint: First use a De Morgan law to show that \( p \lor q \) is logically equivalent to \( \neg (\neg p \land \neg q) \).]

45. Show that \( \neg \) and \( \lor \) form a functionally complete collection of logical operators.

The following exercises involve the logical operators NAND and NOR. The proposition \( p \text{ NAND } q \) is true when either \( p \) or \( q \), or both, are false; and it is false when both \( p \) and \( q \) are true. The proposition \( p \text{ NOR } q \) is true when both \( p \) and \( q \) are false, and it is false otherwise. The propositions \( p \text{ NAND } q \) and \( p \text{ NOR } q \) are denoted by \( p \upharpoonright q \) and \( p \downharpoonright q \), respectively. (The operators \| \) and \( \upharpoonright \) are called the Sheffer stroke and the Peirce arrow after H. M. Sheffer and C. S. Peirce, respectively.)

46. Construct a truth table for the logical operator NAND.
47. Show that \( p \| q \) is logically equivalent to \( \neg (p \land q) \).
48. Construct a truth table for the logical operator NOR.
49. Show that \( p \downharpoonright q \) is logically equivalent to \( \neg (p \lor q) \).
50. In this exercise we will show that \( \{\downharpoonright\} \) is a functionally complete collection of logical operators.
   a) Show that \( p \downharpoonright p \) is logically equivalent to \( \neg p \).
   b) Show that \( (p \downharpoonright q) \downharpoonright (p \downharpoonright q) \) is logically equivalent to \( p \lor q \).
   c) Conclude from parts (a) and (b), and Exercise 49, that \( \{\downharpoonright\} \) is a functionally complete collection of logical operators.

51. Find a compound proposition logically equivalent to \( p \rightarrow q \) using only the logical operator \( \downharpoonright \).
52. Show that \( \{\|\} \) is a functionally complete collection of logical operators.
53. Show that \( p \| q \) and \( q \| p \) are equivalent.
54. Show that \( p \| (q \| r) \) and \( (p \| q) \| r \) are not equivalent, so that the logical operator \( \| \) is not associative.

55. How many different truth tables of compound propositions are there that involve the propositional variables \( p \) and \( q \)?

56. Show that if \( p, q, \) and \( r \) are compound propositions such that \( p \) and \( q \) are logically equivalent and \( q \) and \( r \) are logically equivalent, then \( p \) and \( r \) are logically equivalent.

57. The following sentence is taken from the specification of a telephone system: "If the directory database is opened, then the monitor is put in a closed state, if the system is not in its initial state." This specification is hard to understand because it involves two conditional statements. Find an equivalent, easier-to-understand specification that involves disjunctions and negations but not conditional statements.

58. How many of the disjunctions \( p \lor \neg q, \neg p \lor q, q \lor r, q \lor \neg r, \) and \( \neg q \lor \neg r \) can be made simultaneously true by an assignment of truth values to \( p, q, \) and \( r \)?

59. How many of the disjunctions \( p \lor \neg q \lor s, \neg p \lor \neg q \lor r, \neg p \lor \neg q \lor \neg s, \neg p \lor q \lor \neg s, q \lor r \lor \neg s, q \lor \neg r \lor \neg s, \neg p \lor q \lor \neg s, p \lor r \lor s, \) and \( p \lor r \lor \neg s \) can be made simultaneously true by an assignment of truth values to \( p, q, r, \) and \( s \)?

60. Show that the negation of an unsatisfiable compound proposition is a tautology and the negation of a compound proposition that is a tautology is unsatisfiable.

61. Determine whether each of these compound propositions is satisfiable.
   a) \( (p \lor \neg q) \land (\neg p \lor q) \land (\neg p \lor \neg q) \)
   b) \( (p \rightarrow q) \land (p \rightarrow \neg q) \land (\neg p \rightarrow q) \land (\neg p \rightarrow \neg q) \)
   c) \( (p \leftrightarrow q) \land (\neg p \leftrightarrow q) \)

62. Determine whether each of these compound propositions is satisfiable.
   a) \( (p \lor q \lor \neg r) \land (p \lor \neg q \lor \neg s) \land (\neg p \lor \neg q \lor \neg s) \land (p \lor \neg q \lor \neg s) \)
   b) \( (\neg p \lor \neg q \lor r) \land (\neg p \lor q \lor \neg s) \land (p \lor \neg q \lor \neg s) \land (\neg p \lor r \lor \neg s) \land (p \lor q \lor \neg r) \land (\neg p \lor q \lor \neg r) \land (\neg p \lor q \lor \neg s) \land (\neg p \lor r \lor \neg s) \)

63. Show how the solution of a given \( 4 \times 4 \) Sudoku puzzle can be found by solving a satisfiability problem.

64. Construct a compound proposition that asserts that every cell of a \( 9 \times 9 \) Sudoku puzzle contains at least one number.

65. Explain the steps in the construction of the compound proposition given in the text that asserts that every column of a \( 9 \times 9 \) Sudoku puzzle contains every number.

*66. Explain the steps in the construction of the compound proposition given in the text that asserts that each of the nine \( 3 \times 3 \) blocks of a \( 9 \times 9 \) Sudoku puzzle contains every number.